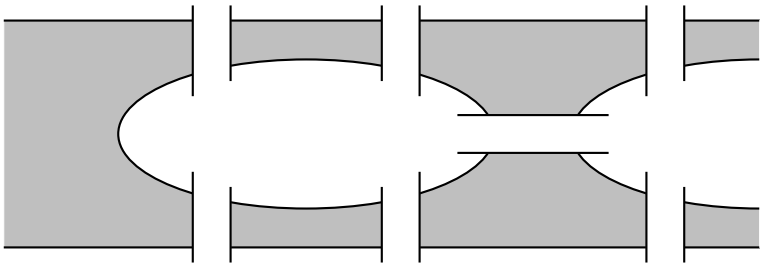
Introduction to Graph Theory.

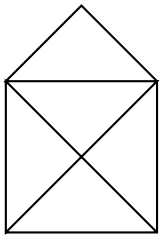
Alain Brobecker - March 2011

Problem #1: The seven bridges of Königsberg.
Can you walk through the city, crossing each bridge once and once only?

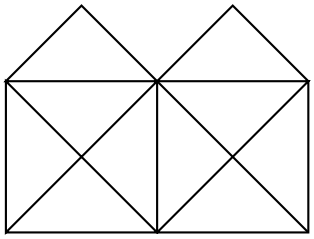


Problem #2: Childhood memories.
Can you draw each of the following objects in one go,
without lifting the pen or running it twice over the same spot?

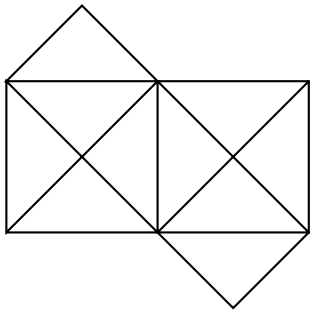
2.a)



2.b)

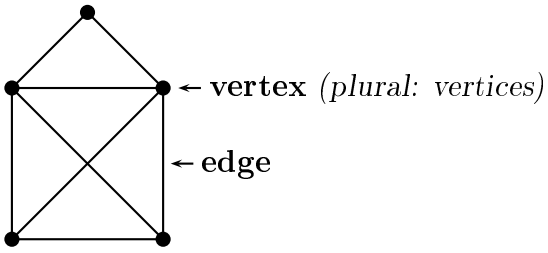


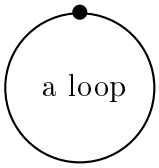
2.c)

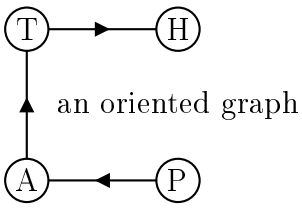


Definitions:

▷ A graph is composed of **vertices** connected together by **edges**.
▷ An edge connects two vertices (or can make a loop on one vertex), and can be oriented.
▷ A vertex can have any number of edges attached to it (even 0). The number of edges connected to a given vertex is called the **order** of this vertex.

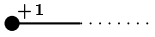


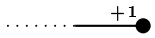


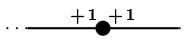


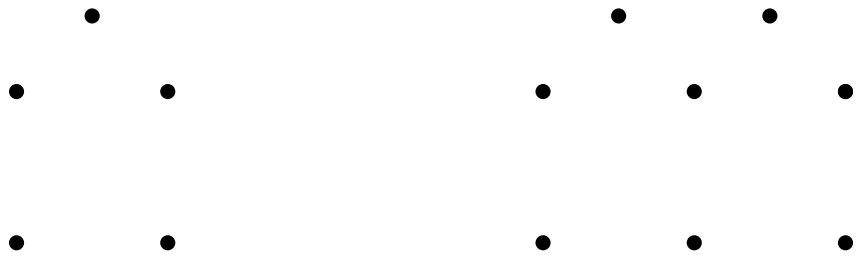
▷ A **path** is a series of vertices correctly connected by edges (P;A;T;H is a path, P;H;A;T is not).
▷ A closed path is a path in which the start vertex is the same as the end vertex.
▷ A path is Eulerian when the path visits each edge of the graph once and once only.
▷ A path is Hamiltonian when the path visits each vertex of the graph once and once only.

Proof that problems #1 and #2.b can not be solved:

A path starts on a vertex, increasing the order of this vertex by 1: 

A path ends on a vertex, increasing the order of this vertex by 1: 

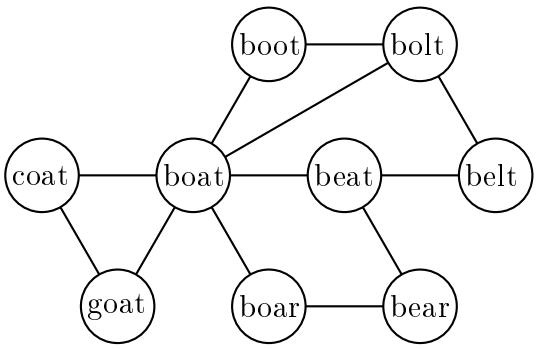
When the path goes through a vertex, the order of this vertex is increased by 2: 



Thus only the start and end vertices of a path can have an odd order. In 1736, Leonhard Euler found this and enunciated that **a path can visit each edge of the graph once and once only if and only if the number of vertices in the graph with an odd order is 0 or 2**.

Applications of graph theory:

A major and difficult problem of the graph theory is to search for the **best** route in a graph. **Best** refers to a mathematical distance defined over the given graph, for example:
▷ roadmap; distance is length of travel.
▷ roadmap; distance is time (taking speed limits and traffic into account).
▷ roadmap; distance is fuel used for the travel.
▷ construction of an airplane; distance is number of days for a given subcontractor to provide pieces.
▷ dictionary; distance is defined as to have similar words near (dictionnary near dictionary).
▷ words of same length; distance is number of modifications of one letter at a time.



Fill in the word ladder:

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coat
....
....
belt
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