Number of ways to place a white and a black king on a $\mathcal{S} \times \mathcal{S}$ chessboard.

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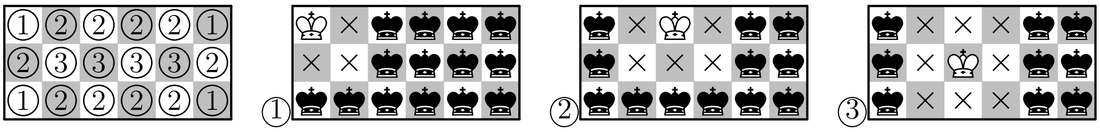
I found this question in Variant Chess (VC) 11, but the formulas, which were corrected in VC 12, were only applying to boards with even sizes and were not explained.

1) Total number of ways to place a white and a black king on a chessboard

We will start by searching the number of ways to place a white and a black king on a rectangular chessboard with $\mathcal{R}$ ranks and $\mathcal{F}$ files (using the same names as in VC and assuming $\mathcal{R} \geqslant 2$, $\mathcal{F} \geqslant 2$ and $\mathcal{R} + \mathcal{F} \geqslant 5$). We have 3 cases, according to the position of the White King (WK):
① in a corner, ② in the inside of a border or ③ inside the board.



① WK is in a corner, 4 possibilities.
Black King (BK) then has $\mathcal{R} \times \mathcal{F} - 4$ possible positions.

③ WK is inside a border, $2 \times (\mathcal{R} - 2) + 2 \times (\mathcal{F} - 2)$ possibilities.
BK then has $\mathcal{R} \times \mathcal{F} - 6$ possibilities.

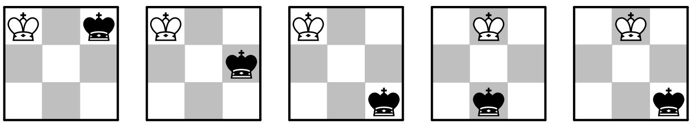
③ WK is inside the half border, $(\mathcal{R} - 2) \times (\mathcal{F} - 2)$ possibilities.
BK then has $\mathcal{R} \times \mathcal{F} - 9$ possibilities.

Total $= 4 \times (\mathcal{R} \times \mathcal{F} - 4) + 2 \times (\mathcal{R} - 2) + 2 \times (\mathcal{F} - 2) \times (\mathcal{R} \times \mathcal{F} - 6) + (\mathcal{R} - 2) \times (\mathcal{F} - 2) \times (\mathcal{R} \times \mathcal{F} - 9)$ Total $= \boxed{\mathcal{R}^2\mathcal{F}^2 - 9\mathcal{R}\mathcal{F} + 6\mathcal{R} + 6\mathcal{F} - 4}$ possibilities to place a white and a black king.

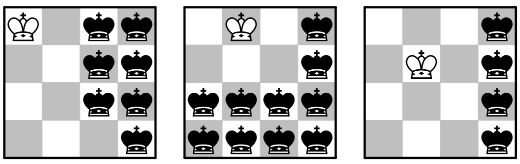
For a square board with a side of size $\mathcal{S} \geqslant 3$ this gives $\boxed{\mathcal{S}^4 - 9\mathcal{S}^2 + 12\mathcal{S} - 4}$ possibilities to place a white and a black king. The resulting sequence is A035286 in the Online Encyclopedia of Integer Sequences (OEIS), and the first values for $\mathcal{S} \geqslant 3$ are 32, 156, 456, 1040, 2040, **3612** ($\mathcal{S} = 8$), 5936...

2) Number of unique ways to place a white and a black king on a square chessboard

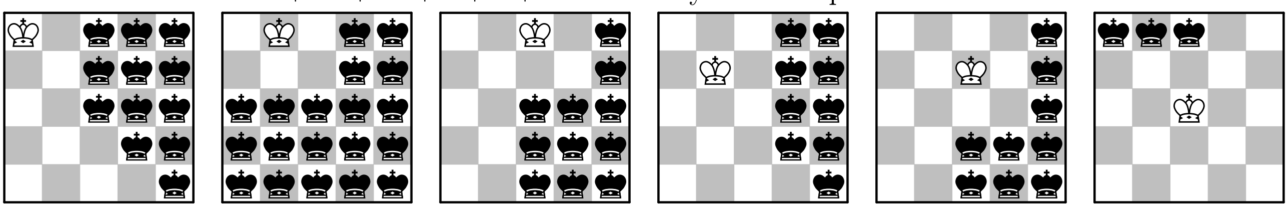
Next, on a square chessboard of size $\mathcal{S} \times \mathcal{S}$ (with $\mathcal{S} \geqslant 3$) we want to know how many **unique** positions there are, this mean not counting twice positions which can be obtained by rotations or symmetries (or a combination) of an already seen position. The chessboard coloring is not taken into account.
For $\mathcal{S} = 3$ we have only 5 really different possibilities:



For $\mathcal{S} = 4$ we have $7 + 10 + 4 = 21$ really different possibilities:



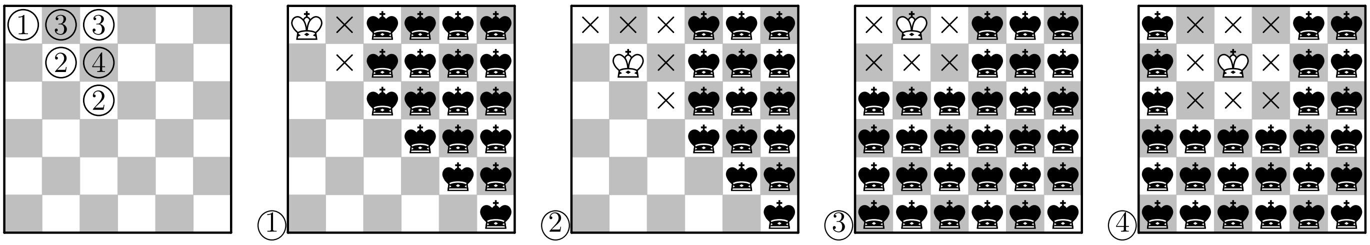
For $\mathcal{S} = 5$ we have $12 + 19 + 11 + 9 + 9 + 3 = 63$ really different possibilities:



We'll have to study separately the cases when $\mathcal{S}$ is even or odd, due to extra symmetries along the central file and central rank when $\mathcal{S}$ is odd.

In both cases I'll first give some diagrams showing in an example the different positions of the WK that we'll have to study, other positions can be obtained from those with rotations and symmetries. Then for each of these positions diagrams will show where the BK can be placed legally.

2.a) First case: $\mathcal{S} = 2 \times s$ is even



① WK is in the corner, 1 possibility.
BK then has $2s + (2s-1) + ... + 1 - 3 = \dfrac{(2s+1) \times 2s}{2} - 3 = 2s^2 + s - 3$ possibilities.

② WK is inside the half diagonal, $s - 1$ possibilities.
BK then has $2s + (2s-1) + ... + 1 - 6 = 2s^2 + s - 6$ possibilities.

③ WK is inside the half border, $s - 1$ possibilities.
BK then has $4s^2 - 6$ possibilities.

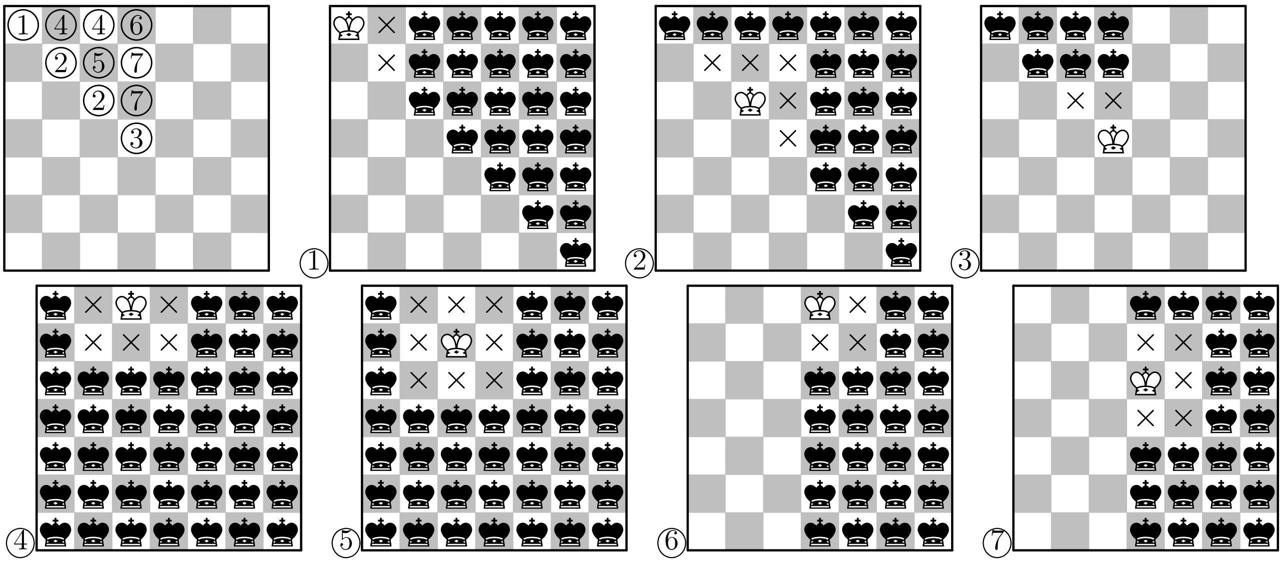
④ WK is inside the board, $(s-2) + (s-3) + ... + 1 = \dfrac{(s-2) \times (s-1)}{2} = \dfrac{s^2 - 3s + 2}{2}$ possibilities.
BK then has $4s^2 - 9$ possibilities.

Total $= 2s^2 + s - 3 + (s-1) \times (2s^2 + s - 6) + (s-1) \times (4s^2 - 6) + \dfrac{s^2 - 3s + 2}{2} \times (4s^2 - 9)$
$= 2s^2 + s - 3 + 2s^3 + s^2 - 6s - 2s^2 - s + 6 + 4s^3 - 6s - 4s^2 + 6 + 2s^4 - \frac{9}{2}s^2 - 6s^3 + \frac{27}{2}s + 4s^2 - 9$
$= 2s^4 + 2s^3 + 4s^3 - 6s^3 + 2s^2 + s^2 - 2s^2 - 4s^2 - \frac{9}{2}s^2 + 4s^2 + s - 6s - s - 6s + \frac{27}{2}s - 3 + 6 + 6 - 9$
$= 2s^4 - \dfrac{7}{2}s^2 + \dfrac{3}{2}s$ different possibilities to place a white and a black king.

Knowing that $s = \dfrac{\mathcal{S}}{2}$ this finally gives $\boxed{\dfrac{\mathcal{S}^4}{8} - \dfrac{7\mathcal{S}^2}{8} + \dfrac{3\mathcal{S}}{4}}$ different possibilities to place a white and a black king on a $\mathcal{S} \times \mathcal{S}$ chessboard.

This is the same formula as the one given in Variant Chess by **Roy Talbot** and **Paul Byway** in 1993, and confirmed by George Jelliss. They gave the result only for a square chessboard of size $\mathcal{S} = 8$, which has 462 unique positions for a white and a black king. Another quick verification is for $\mathcal{S} = 4$, the formula gives 21 which is also the count given in the example on page 1.
An array with more results is given when the second case will be studied.

2.b) Second case: $\mathcal{S} = 2 \times s + 1$ is odd



① WK is in a corner, 1 possibility.
BK then has $(2s+1) + 2s + ... + 1 - 3 = \dfrac{(2s+2)\times(2s+1)}{2} - 3 = 2s^2 + 3s - 2$ possibilities.

② WK is inside the half diagonal, $s-1$ possibilities.
BK then has $(2s+1) + 2s + ... + 1 - 6 = 2s^2 + 3s - 5$ possibilities.

③ WK is in the center of the board, 1 possibility.
BK then has $(s+1) + s + ... + 1 - 3 = \dfrac{s^2+3s}{2} - 2$ possibilities.

④ WK is inside the half border, $s-1$ possibilities.
BK then has $(2s+1)^2 - 6 = 4s^2 + 4s - 5$ possibilities.

⑤ WK is inside the board, $(s-2) + (s-3) + ... + 1 = \dfrac{(s-2)\times(s-1)}{2} = \dfrac{s^2-3s+2}{2}$ possibilities.
BK then has $(2s+1)^2 - 9 = 4s^2 + 4s - 8$ possibilities.

⑥ WK is in the middle of a border, 1 possibility.
BK then has $(2s+1)\times(s+1) - 4 = 2s^2 + 3s - 3$ possibilities.

⑦ WK is inside the central half file, $s-1$ possibilities.
BK then has $(2s+1)\times(s+1) - 6 = 2s^2 + 3s - 5$ possibilities.

Total $= 4s^2 + 6s - 5 + (s-1)\times[8s^2 + 10s - 15] + \frac{s^2+3s}{2} - 2 + (s^2 - 3s + 2)\times(2s^2 + 2s - 4)$
$= 4s^2 + 6s - 5 + 8s^3 + 10s^2 - 15s - 8s^2 - 10s + 15 + \frac{s^2+3s}{2} - 2 + 2s^4 + 2s^3 - 4s^2 - 6s^3 - 6s^2 + 12s + 4s^2 + 4s - 8$
$= 2s^4 + 4s^3 + \dfrac{s^2}{2} - \dfrac{3s}{2}$

Knowing that $s = \dfrac{\mathcal{S}-1}{2}$ this finally gives $\boxed{\dfrac{\mathcal{S}^4}{8} - \dfrac{5\mathcal{S}^2}{8} + \dfrac{1}{2}}$ different possibilities to place a white and a black king on a $\mathcal{S}\times\mathcal{S}$ chessboard.

The formulas are the same as given by **Nathan L. Skirrow** in 2022 for sequence A357723. So the number of different ways to place a white and a black king on a $\mathcal{S}\times\mathcal{S}$ chessboard is:

$\mathcal{S}$	3	4	5	6	7	8	9	10	11	12	13	14
total A035286	32	156	456	1040	2040	**3612**	5936	9216	13680	19580	27192	36816
unique A357723	5	21	63	135	270	**462**	770	1170	1755	2475	3465	4641