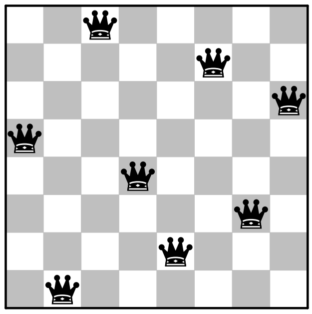
Non-Attacking Rooks on HexHex and Triangular boards - *Alain Brobecker*

0) Square boards

A classical chess problem is to place 8 queens on an 8×8 chessboard so that they don't attack each other. An extension is to find in how many ways you can place N non-attacking queens on an $N \times N$ board made of squares. I give an example solution and the table below gives results for this question and links for further investigation.
Of course you can never place more queens than N, since you'll have at most 1 queen per row or per column. In fact for $N \in [2;3]$ you'll in fact place less queens than N.

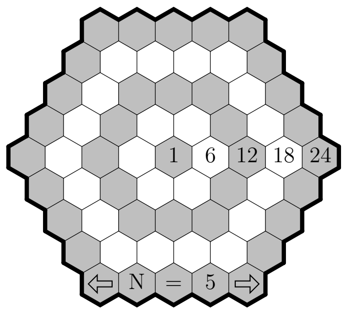


side of the board	1	2	3	4	5	6	7	8	9	N
cells in board	1	4	9	16	25	36	49	64	81	N^2
max queens	1	**1***	**2***	4	5	6	7	8	9	$N^\star$
solutions	1	$1^{\star\star}$	$8^{\star\star}$	2	10	4	40	92	352	A000170
fundamental sols	1	$1^{\star\star}$	$1^{\star\star}$	1	2	1	6	12	46	A002562

* : of course the values in bold are not N
** : should be 0 if we stick to N queens

1) HexHex boards

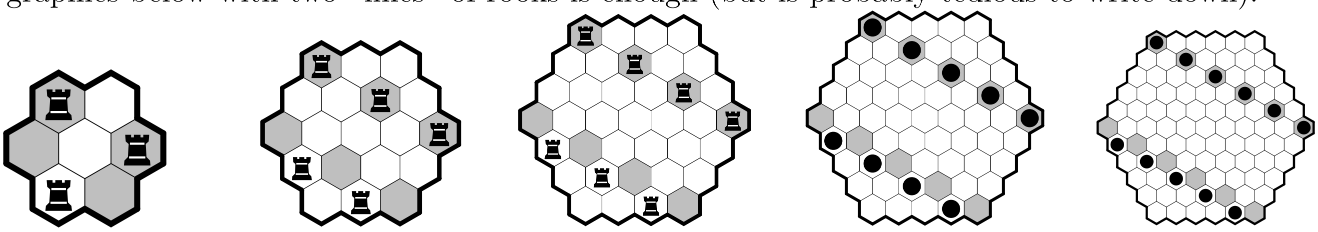
The same question can be asked on an hexagonal board made of hexagons (an HexHex board, see aside).
Due to the definition of neighbours which is way simpler on such a board (we don't have to care about the difference between orthogonal and diagonal neighbours), we can consider that queens and rooks are the same piece.



An HexHex board with side N has $\boxed{1 + 3 \times (N-1) \times N}$ hexagons.

$$\left(\textit{because } 1 + 6 + 12 + 18 + ... + 6 \times (N-1) = 1 + 6 \times \sum_{k=1}^{N-1} k = 1 + 6 \times \frac{(N-1) \times N}{2} \right)$$

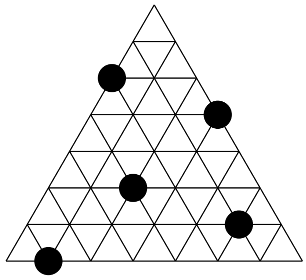
Also we know that we'll be able to place at most $2N-1$ rooks on such a board since this is the number of lines in each of the 3 directions. To prove that $2n-1$ rooks can be placed, I think the graphics below with two "lines" of rooks is enough (but is probably tedious to write down).



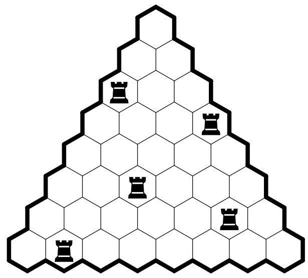
Using a homemade program I found the results below. The number of solutions found is the same as the one given by Václav Kotěšovec, which is always a good sign. The number of fundamental solutions is found by excluding identical rooks placements after rotations or relections leaving the board inchanged. In annex A you will find graphical output for some fundamental solutions.

side of the board	1	2	3	4	5	6	7	8	9	N
cells in board	1	7	19	37	61	91	127	169	217	$1 + 3 \times (N-1) \times N$
max rooks	1	3	5	7	9	11	13	15	17	$2N-1$
solutions	1	2	6	28	244	2544	35600	659632	15106128	A002047
fundamental sols	1	1	1	5	29	224	3012	55200	1259794	A309260

2) Triangular boards



And the same question can also be asked on triangular boards. The left diagram shows a triangular board with a triangular net, which is **equivalent** to the right diagram showing a triangular board made of hexagons. Movements of rooks (or queens) transpose directly from one board to another.



It is clear that the side of the board made with hexagons on the right has size 8, but it is less clear on the board made of triangles. To clarify it, we'll define the size of a side as the number of places a piece can stand on this side, so 8 on the left picture (and you have 7 inbetween segments).

A triangular board with side N has $1+2+3+...+N = \sum_{k=1}^{N} k = \dfrac{N \times (N+1)}{2}$ spaces.

Since everything is equivalent to a board made with hexagons, I was able to use almost the same program as before to do the computations and found the following results:

side of the board	1	2	3	4	5	6	7	8	9	N
cells in board	1	3	6	10	15	21	28	36	45	$\frac{N \times (N+1)}{2}$
max rooks	1	1	2	3	3	4	5	5	6	A004396***
solutions	1	3	3	2	23	18	6	270	166	A289893****
fundamental sols	1	1	1	1	5	3	1	45	29	A350041

side of the board	10	11	12	13	14	15	16	N
cells in board	55	66	78	91	105	120	136	$\frac{N \times (N+1)}{2}$
max rooks	7	7	8	9	9	10	11	A004396***
solutions	28	4842	2532	244	120052	49620	2544	A289893****
fundamental sols	6	807	422	46	20022	8270	424	A350041

In Annex B you will find graphical output of fundamental solutions for $N \in [1;10]$.

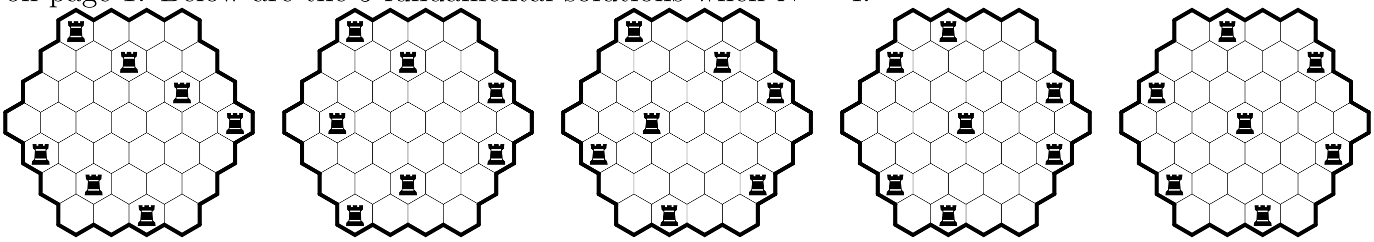
A004396***: Maximal number of points on a triangular grid of edge length n-1 with no 2 points on same row, column, or diagonal. See Problem 252 in The Inquisitive Problem Solver. - R. K. Guy [Comment revised by N. J. A. Sloane, Jul 01 2016]

A289893****: Number of maximum independent vertex sets in the n-triangular honeycomb queen graph.

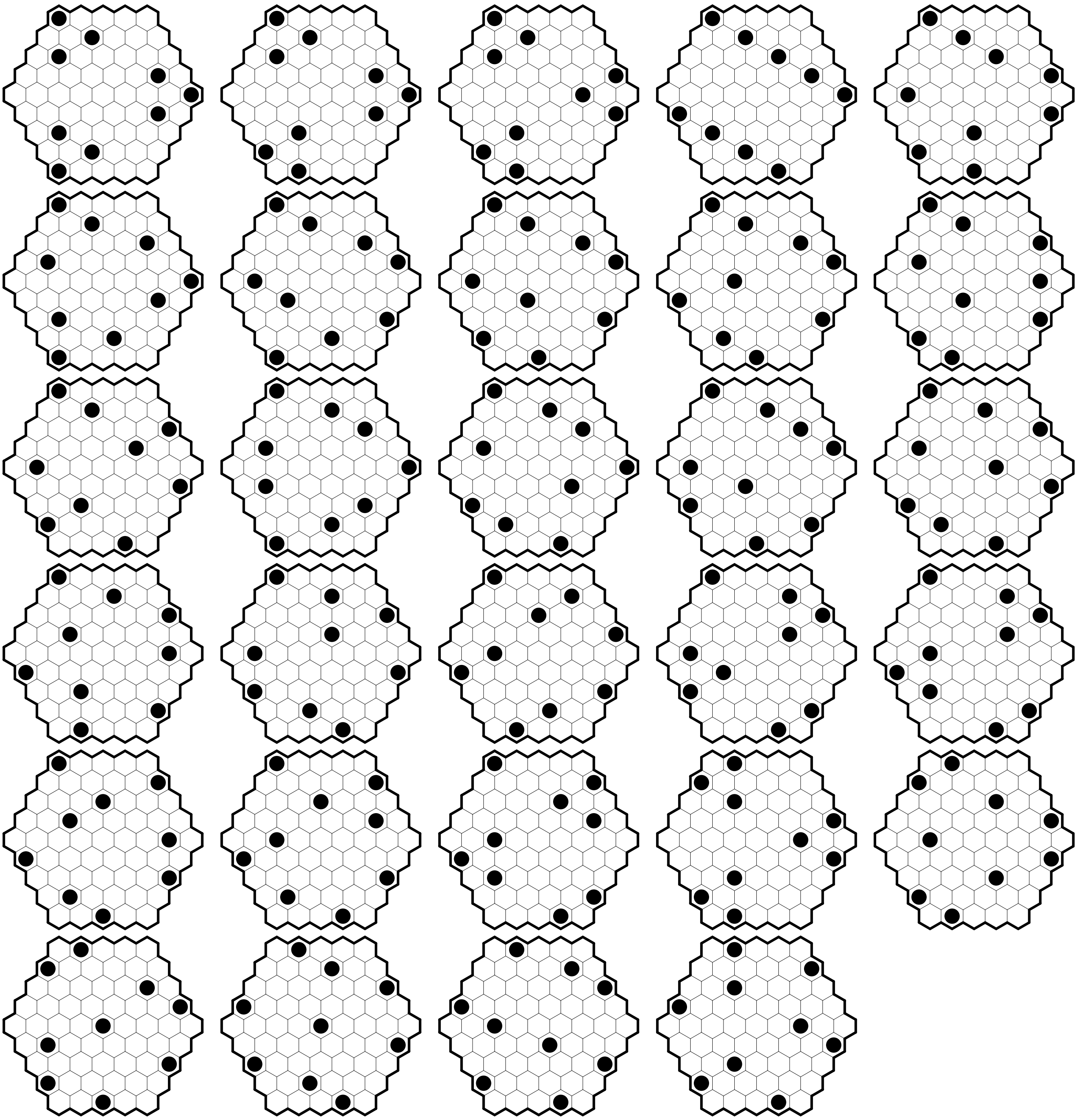
TO DO: proof that max number of rooks is the serie constituted of two odd numbers followed by an even number and so on... Annex B showing all the fundamental solutions might be helpful!

Annex A: HexHex boards

The fundamental solutions when side of the board is $1 \leqslant N \leqslant 3$ are either obvious or already given on page 1. Below are the 5 fundamental solutions when $N = 4$:



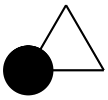
Below are the 29 fundamental solutions when $N = 5$:

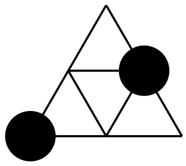


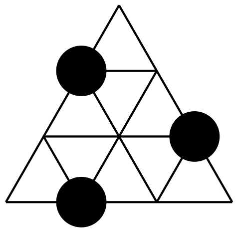
Annex B: Triangular boards

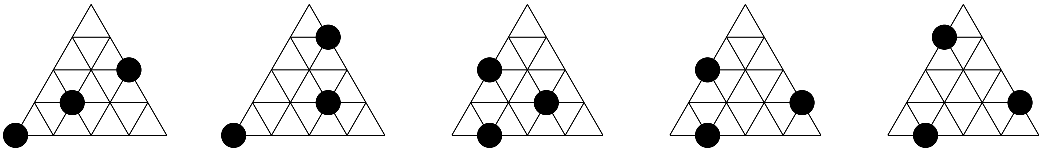
The fundamental solution when side of the board is $N = 1$ is obvious, below are all fundamental solutions for $1 \leqslant N \leqslant 10$:

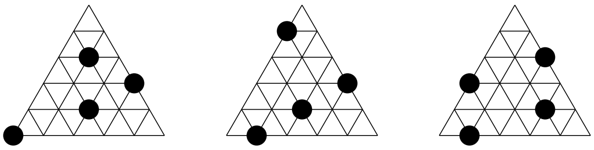
$N = 1$

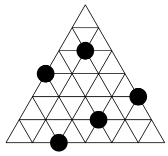

$N = 2$


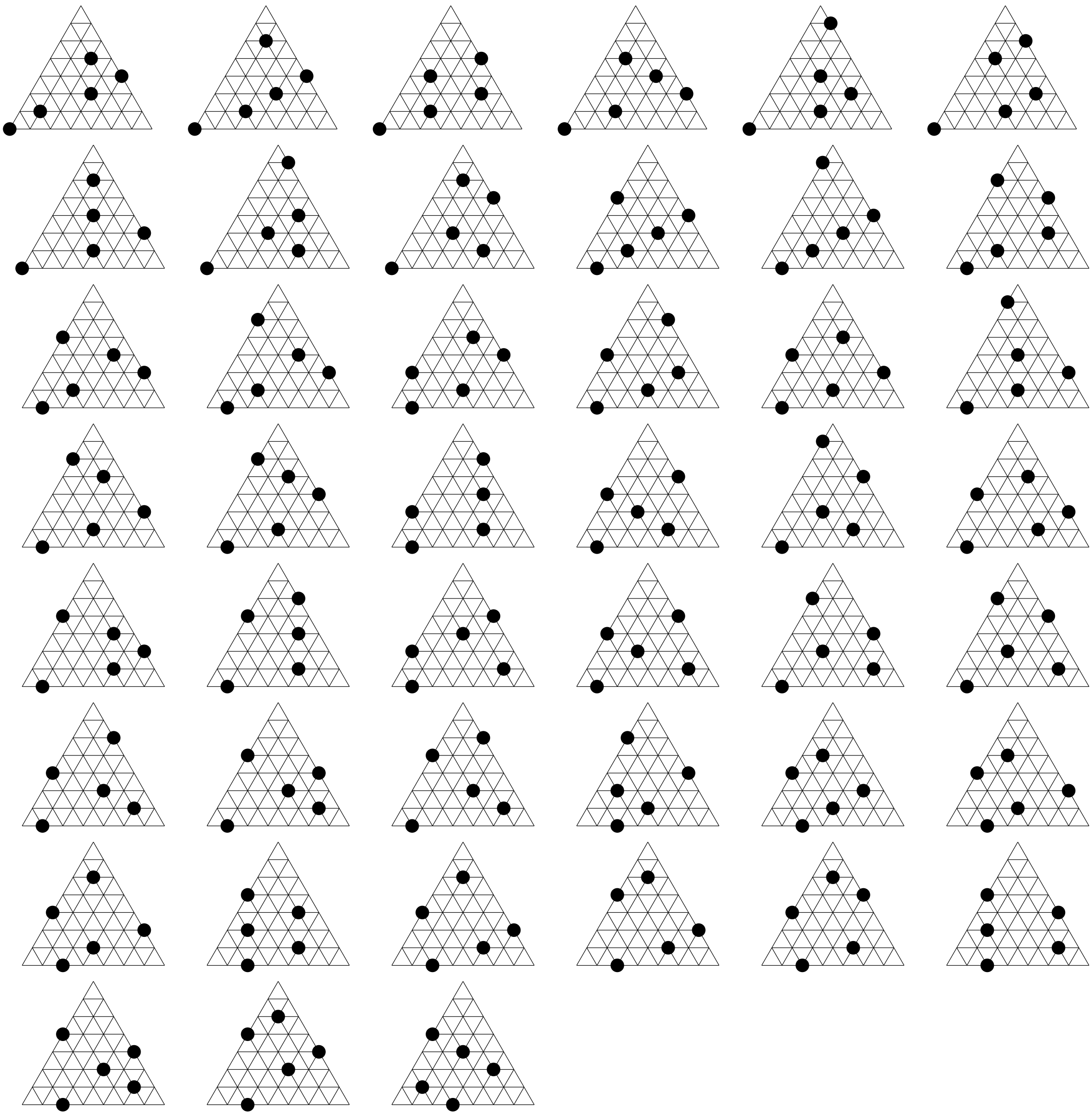
$N = 3$


$N = 4$


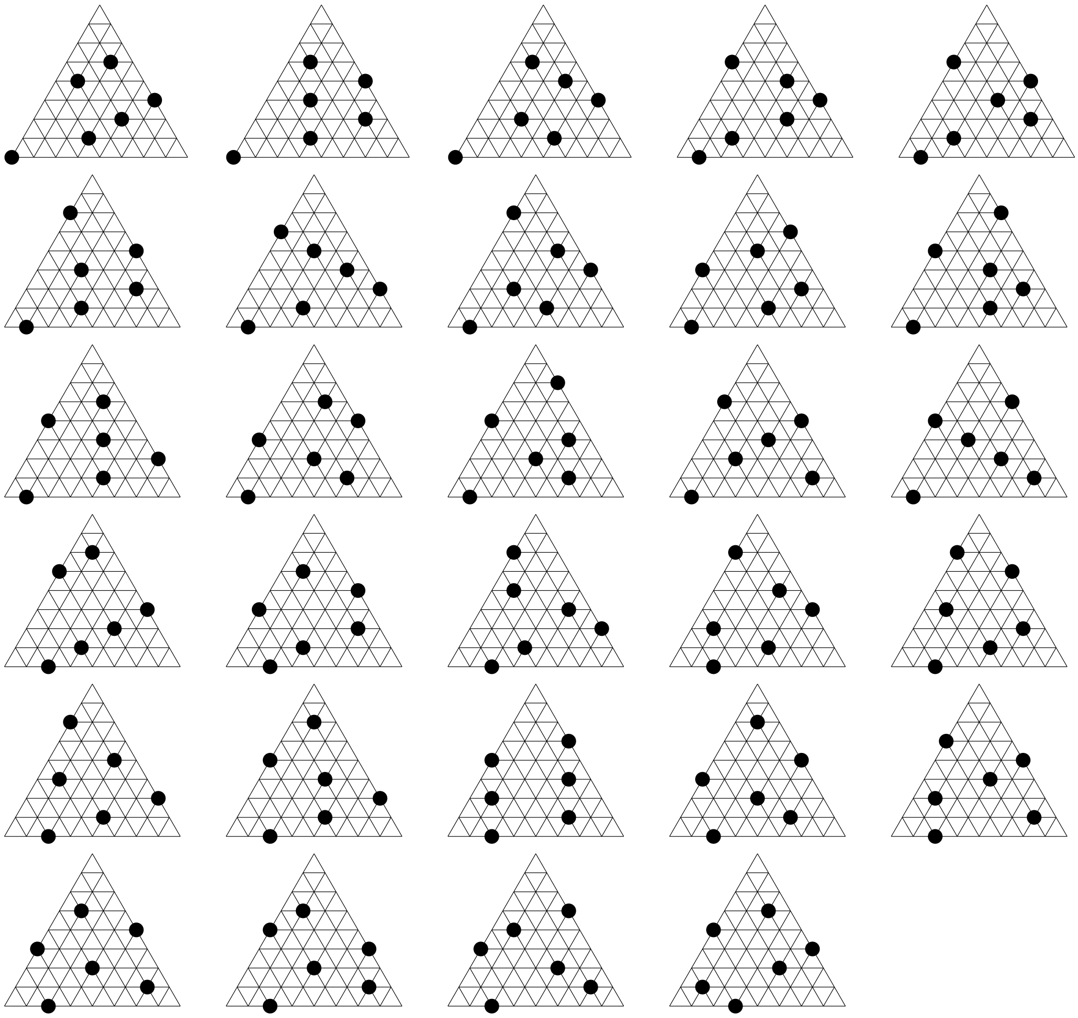
$N = 5$


$N = 6$


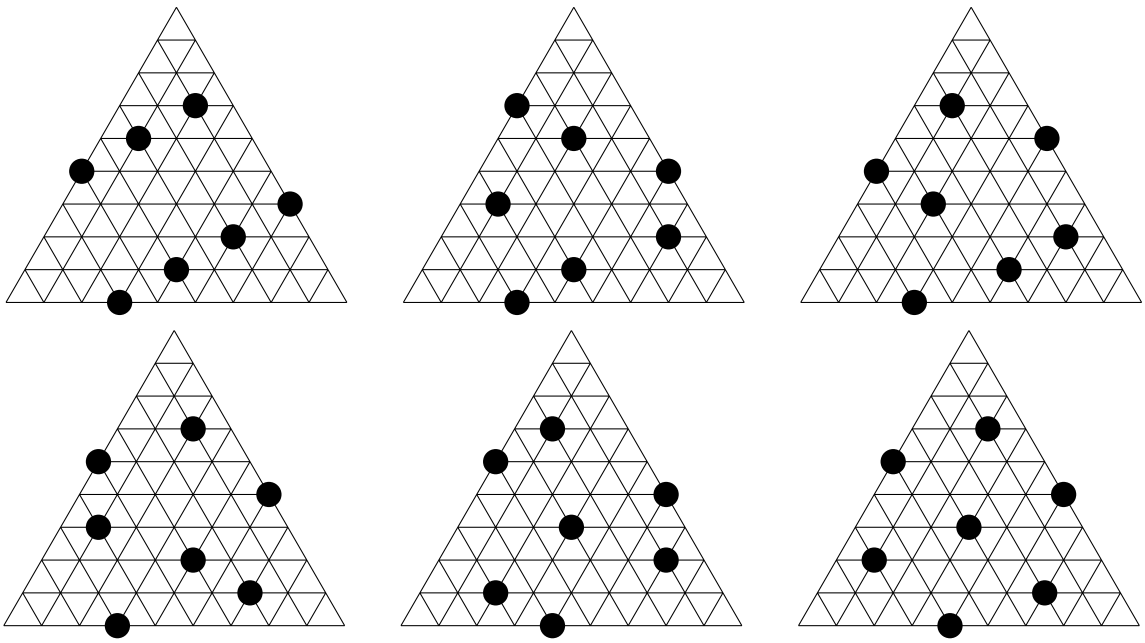
$N = 7$


$N = 8$


$N = 9$

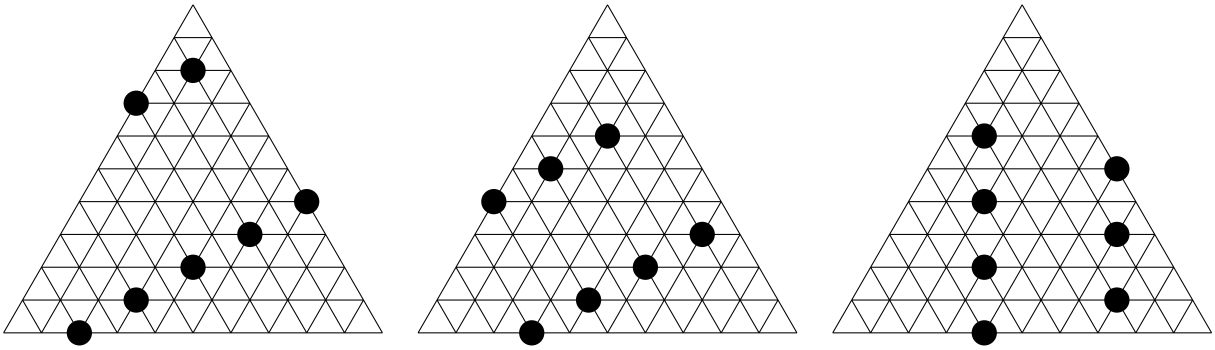


$N = 10$

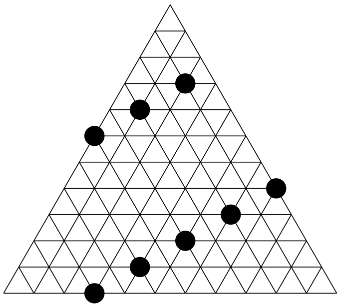


Václav Kotěšovec points out that the **relative** position of rooks in 3^{rd} and 4^th diagrams are exactly the same (although they are not at the same places on the board after rotation and/or mirroring which was what I checked) and thus he considers they could be considered identical. Thus his calculations gave other values for $N = 10$, $N = 13$ and $N = 16$... and can be found in A350191.

$N = 11$



$N = 12$



$N = 13$