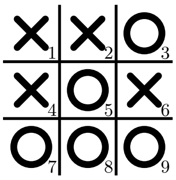
Tic-tac-toe and retrograde analysis

Alain Brobecker - september 2007 - http://abrobecker.free.fr/

(important note: it's not always the same symbol that starts a game in the problems shown)

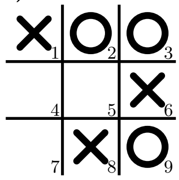


Last move?

Retrograde analysis is possible in the ultra simple tic-tac-toe game. There exist only 6 "problems" of type "last move?", if we forget about symmetrical positions. The diagram aside shows an example, but everyone will admit it's very easy and giving the solution is superfluous.

The website of Joe Kisenwether (www.geocities.com/joe_kisenwether/Retro.html) is dedicated to non chess retrograde analysis and shows something more challenging. The following problem was composed by **Les Marvin**:

A) Les Marvin

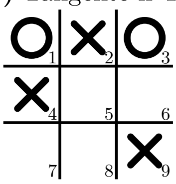


" In this partially completed tic-tac-toe game, **both players were experts** (neither one ever afforded the other an opportunity to force a win). What were the first and the last moves played? "
(This also requires to know who played first. Now C+, ie verified by computer.)

Here's the solution as written by Les Marvin:
A) If we number the squares like a push button phone, the first move was at 2, the last at 6.
If O played last, then X can force a win by playing at 4. Since O is an expert, he would not have allowed this. Thus it was X that played last. Since each player has moved three times, this means that O went first.
The only non-losing response to a corner opening is to play in the middle. Since the middle is not taken, the opening move could not have been in a corner and must therefore have been at 2.
With that established, trial and error reveals only two possible orders in which the moves could have been played with neither player giving the other a chance to force the win: 2,1,9,8,3,6 or 2,8,3,1,9,6
(This is wrong the number of possible orders is 3, the third one being 2,8,9,1,3,6).

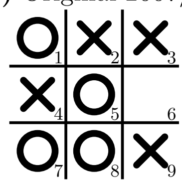
Using a homebrew program, i made some extensive searching in the tic-tac-toe game tree, and was able to find a few more problems, still assuming that **both players are experts** (neither one ever afforded the other an opportunity to force a win). As far as i know **B)** is the longest uniquely realisable game under this condition.

B) Tangente n°122



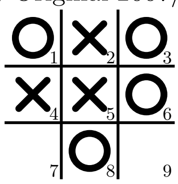
Find the game. (C+)

C) Original 2007/09



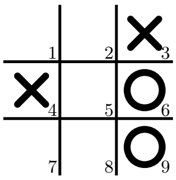
Last move? (C+)

D) Original 2007/09



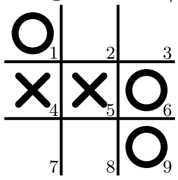
Last 4 moves? (C+)

E) Variant Chess n°56



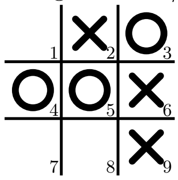
Find the game. (C+)

F) Original 2007/09



Last 2 moves? (C+)

G) Original 2007/09



Last 3 moves? (C+)

The partial game trees used to find the problems are provided on page 3 and 4. On page 5 and 6 are the partial game trees for the "misère" version of tic-tac-toe game, where you must force your opponent to align 3 symbols. As in the regular version, the game is a draw, but it may prove a challenge when played for the first time.

The table on next page shows some statistics about the tic-tac-toe game tree. Symmetrical or rotated positions were discarded (but not symmetrical or rotated games). "won games/pos." stands for games/positions in which an alignment of 3 symbols has just occured. A won game/position is not to be confused with a winning game, where the game/position is not yet finished but one player has a winning procedure. "draw pos." is for positions in which no player has a winning procedure (misère version in last column). We can calculate the number of winning positions by subtracting "won pos." and "draw pos." to "≠ positions".

So the full game tree contains 1 + 3 + 12 + 38 + ... + 15 = 765 different positions.
The number of drawn games is 127872 - 81792 = 46080 games.
Then the total number of games is the sum of all the games won at different plies plus the drawn games, so 1440 + 5328 + ... + 81792 + 46080 = 255168 games.

ply	games	won games	≠ positions	won pos.	draw pos.	draw pos. (misère)
0	1	0	1	0	1	1
1	9	0	3	0	3	1
2	72	0	12	0	5	7
3	504	0	38	0	21	6
4	3024	0	108	0	18	31
5	15120	1440	174	21	35	20
6	54720	5328	204	21	27	43
7	148176	47952	153	58	27	17
8	200448	72576	57	23	11	11
9	127872	81792	15	12	3	3

Solutions of the problems:

The squares will be numbered like a push button phone:

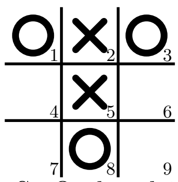
1	2	3
4	5	6
7	8	9

Then we'll use the following rules:
Rule 1: for a given position (with an even number of symbols), if a player can force a win, then it's his opponent to move since no player would have allowed the other one to reach a winning position.
Rule 2: if first player opens in the centre, second player must play in a corner.
Rule 3: if first player opens in a corner, second player must play in the centre.
Rule 4: if first player opens on an edge, second player must play in the centre or in an adjacent corner or on the opposite edge.

B) X just played and his last move can't have been 9 (or 4), since 5 (respectively 8) would have won. So X's last move is 2. X's first move can't have been 9 since O would have played 5 according to rule 3. So X started with 4 and rule 4 implies that O answered with 1. So we can conclude that game was 4,1,9,3,2.

C) X to play would win by playing 8, so rule 1 implies X played last. X has not played 2 or 4 since playing this pawn to 6 instead would have won. If X played 9, then by removing whatever O pawn we still have an open alignment of two Os, so O could have won, and our hypothesis is wrong. Thus the last move was X playing at 3. Two example games could be: 8,9,1,2,5,4,7,3 and 5,9,8,2,1,4,7,3. (No the 4,7,3 sequence at the end is not forced).

D) O just played 6 since the other possibilities could have been replaced by the winning move on 9. X's last move can't have been 2 since 6 would have won. It can't be 5 either, because whatever O move we retract, X had another winning move. So X's last move is 4. Before this two plies, the symmetrical position was:



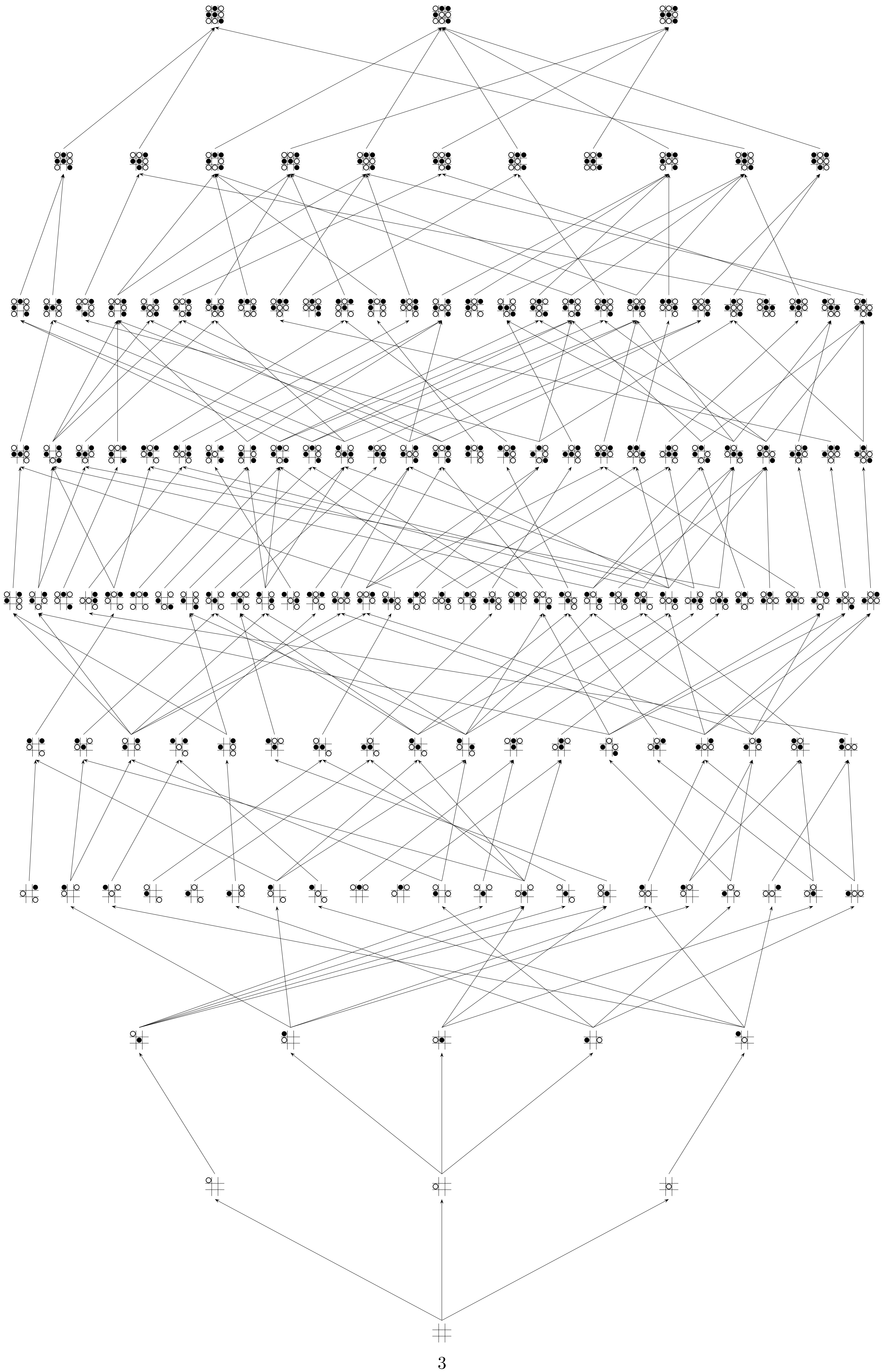
We know O can't have just played 1 (or 3) because 9 (respectively 7) would have won instead. So O played 8 and we also know O started in a corner. Rule 3 implies X answer to O's move in a corner was in the centre, then O played the other corner and X played 2. The two possible games are 1,5,3,2,8,4,6 and 3,5,1,2,8,4,6. So the last 4 moves are 2,8,4,6.

E) X to play could force a win by playing 1, rule 1 implies that X played last. So O started and we know from rule 3 that he played 6 then 9. After 6,3,9 X could force a win by 2,1,5,... so it was not 3 that was played on second move, and thus the game was 6,4,9,3.

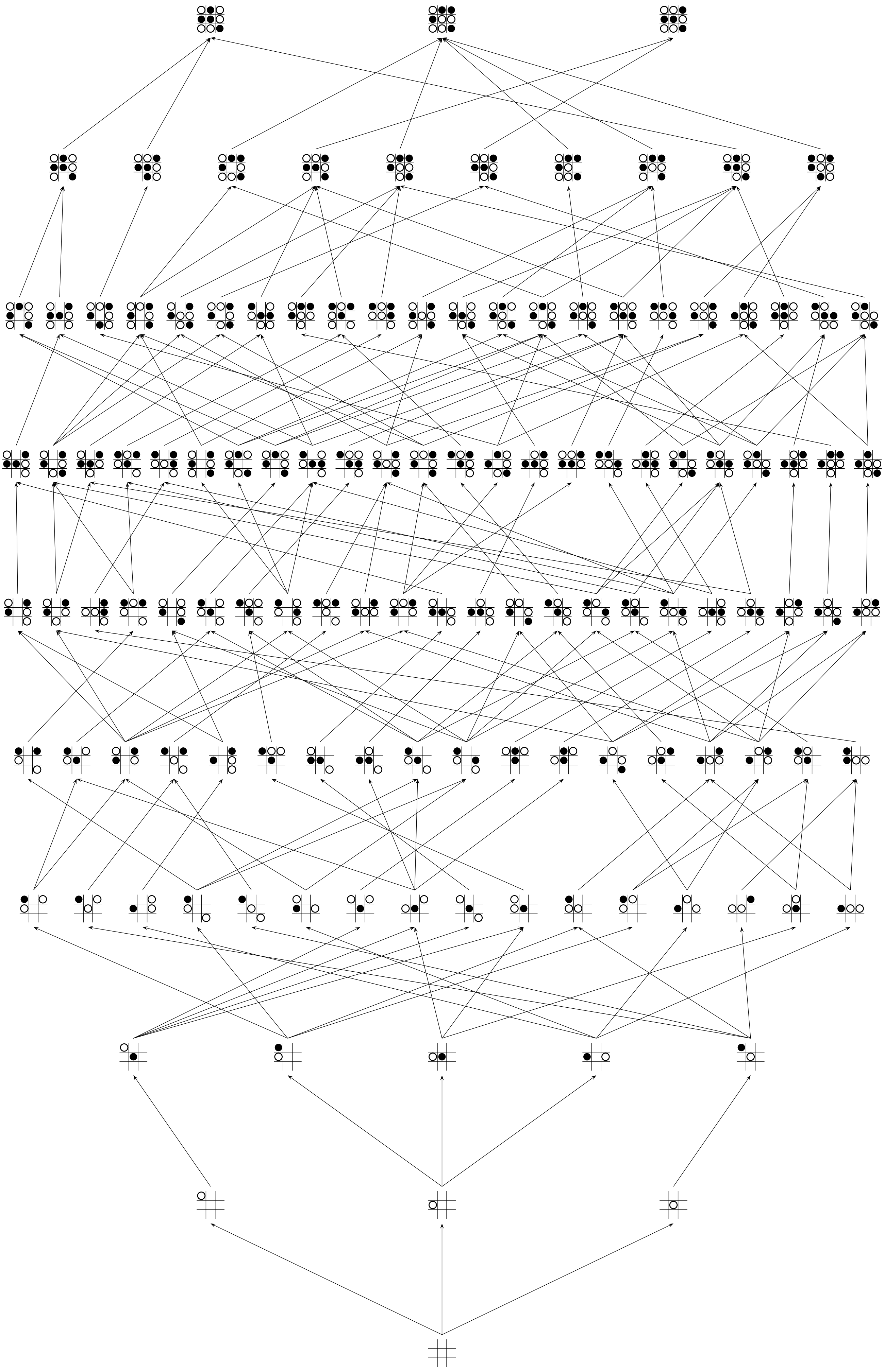
F) If last move was 1, O could have won by playing 3 instead. If last move was 9, O could have forced a win by playing 3 instead. So last move was 6. Also we deduce that O started in a corner, and rule 3 implies that X answered in the centre. So X's last move was 4. The game was 1,5,9,4,6 or 9,5,1,4,6.

G) O to play could win by playing 7, so rule 1 implies that O played last. O's last move wasn't 4 or 5 because playing 7 instead would have won directly or forced a win (respectively). So O's last move was 3. X's previous move wasn't 2 or 9 since playing in 3 would have won directly or forced a win (respectively). So X's previous move was 6. Now, if O's previous move was 5 then whatever was X's first move, 2 or 9, the response 4 was incorrect according to rules 3 and 4. So O's previous move was 4. The game was 2,5,9,4,6,3 or 9,5,2,4,6,3.

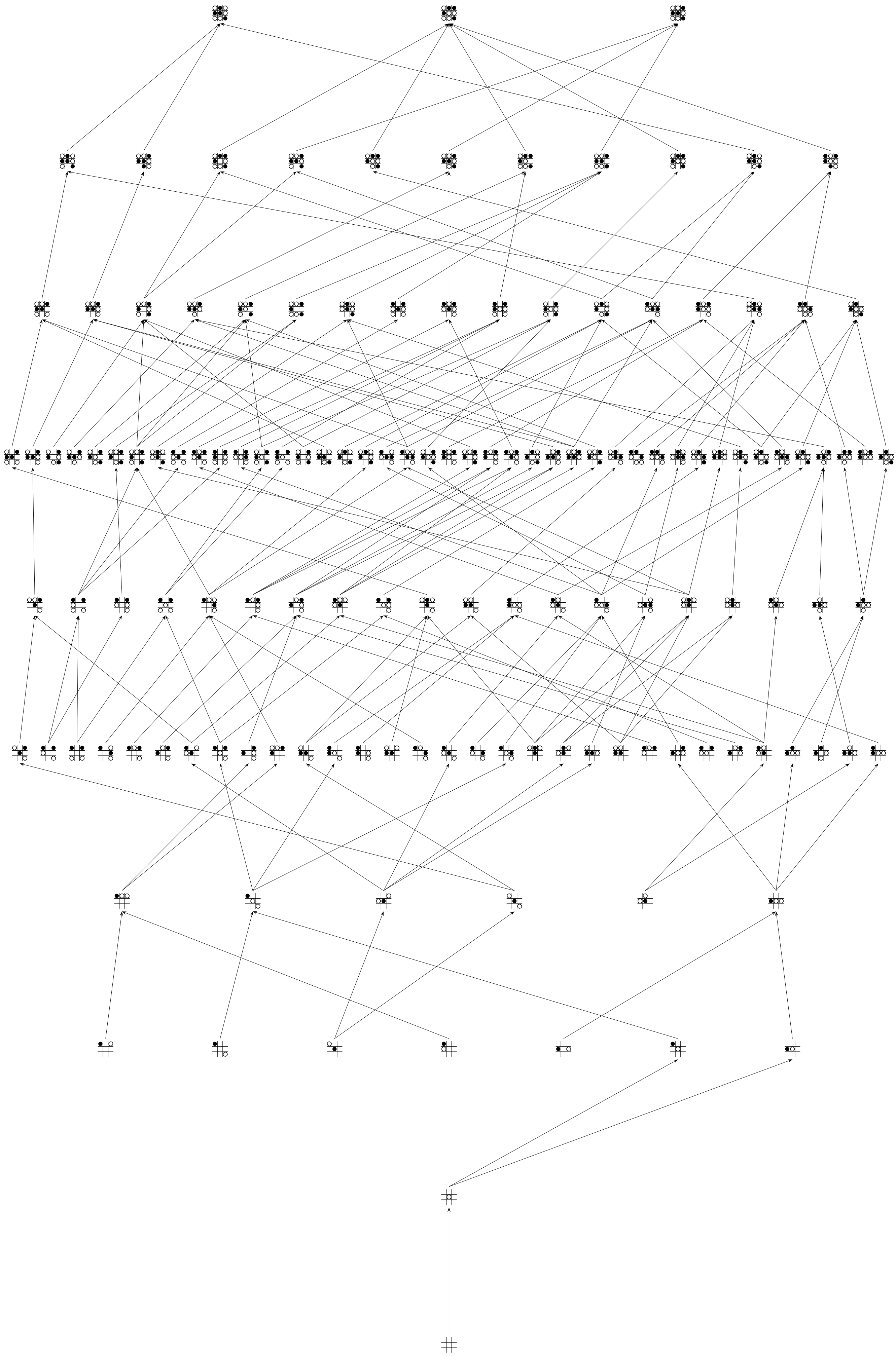
All drawn positions of tic-tac-toe



All positions that can be reached with perfect play in tic-tac-toe



All drawn positions of "misère" tic-tac-toe



All positions that can be reached with perfect play in "misère" tic-tac-toe